Acta Mathematica Sinica, English Series
Mar., 2013, Vol. 29, No. 3, pp. 489–504
Published online: October 26, 2012
DOI: 10.1007/s10114-012-1383-6
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Relative Syzygies and Grade of Modules

Zeng Feng LIU **Zhao Yong HUANG**[1)]
Department of Mathematics, Nanjing University, Nanjing 210093*, P. R. China*
E-mail: *zengfeng*109*@*163*.com* *huangzy@nju.edu.cn*

Abstract Recently, Takahashi established a new approximation theory for finitely generated modules over commutative Noetherian rings, which unifies the spherical approximation theorem due to Auslander and Bridger and the Cohen–Macaulay approximation theorem due to Auslander and Buchweitz. In this paper we generalize these results to much more general case over non-commutative rings. As an application, we establish a relation between the injective dimension of a generalized tilting module ω and the finitistic dimension with respect to ω.

Keywords n-$\mathscr{C}$-Syzygy modules, n-ω-torsionfree modules, approximation presentations, grade

MR(2010) Subject Classification 16E05, 16E10, 16E30

1 Introduction

In 1969, Auslander and Bridger introduced in [1] the notion of n-torsionfree modules, which is a special case of n-syzygy modules. Then they obtained an equivalent characterization for an n-syzygy module of a finitely generated module M being n-torsionfree in terms of a spherical approximation presentation of M over a commutative Noetherian ring. On the other hand, Auslander and Buchweitz proved in [2] that there exists a Cohen–Macaulay approximation presentation for any finitely generated module over a commutative Cohen–Macaulay local ring with the canonical module. Recently, Takahashi unified these two approximation theorems and established a new approximation theory for finitely generated modules over commutative Noetherian rings as follows.

Theorem 1.1 ([3, Theorem A]) *Let R be a commutative Noetherian ring, and let M and C be finitely generated R-modules and n a positive integer. If C is n-semidualizing, then the following statements are equivalent.*

(1) *Any n-syzygy of M is n-C-torsionfree.*

(2) *There exists an exact sequence $0 \to Y \to X \to M \to 0$ of finitely generated R-modules such that $\mathrm{Ext}^i_R(X, C) = 0$ for any $1 \le i \le n$, and there exists an exact sequence $0 \to C_{n-1} \to \cdots \to C_1 \to C_0 \to Y \to 0$ of R-modules with each C_i isomorphic to a direct summand of a finite direct sum of copies of C.*

Received July 21, 2011, accepted March 9, 2012
Supported by the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20100091110034), National Natural Science Foundation of China (Grant No. 11171142), Natural Science Foundation of Jiangsu Province of China (Grant Nos. BK2010047, BK2010007) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions
1) Corresponding author

Let R and S be rings. We use $\operatorname{mod} R$ (resp. $\operatorname{mod} S^{\mathrm{op}}$) to denote the category of finitely generated left R-modules (resp. right S-modules). Let ${}_R\omega_S$ be an (R,S)-bimodule with ${}_R\omega$ in $\operatorname{mod} R$ and ω_S in $\operatorname{mod} S^{\mathrm{op}}$. We use $\operatorname{add}_R \omega$ (resp. $\operatorname{add} \omega_S$) to denote the full subcategory of $\operatorname{mod} R$ (resp. $\operatorname{mod} S^{\mathrm{op}}$) consisting of all modules isomorphic to the direct summands of finite direct sums of copies of ${}_R\omega$ (resp. ω_S). We denote either $\operatorname{Hom}_R(-,{}_R\omega_S)$ or $\operatorname{Hom}_{S^{\mathrm{op}}}(-,{}_R\omega_S)$ by $(-)^{\omega}$.

For any $n \geq 1$, an (R,S)-bimodule ${}_R\omega_S$ with ${}_R\omega \in \operatorname{mod} R$ and $\omega_S \in \operatorname{mod} S^{\mathrm{op}}$ is called *faithfully balanced and n-selforthogonal* if the following conditions are satisfied: (1) $R = \operatorname{End}(\omega_S)$ and $S = \operatorname{End}({}_R\omega)$, and (2) $\operatorname{Ext}_R^i({}_R\omega, {}_R\omega) = 0 = \operatorname{Ext}_{S^{\mathrm{op}}}^i(\omega_S, \omega_S)$ for any $1 \leq i \leq n$. When $R\,(=S)$ is commutative, the notion of faithfully balanced and n-selforthogonal bimodules coincides with that of n-semidualizing bimodules in [3]. If R is a left Noetherian ring, S is a right Noetherian ring and ${}_R\omega_S$ is a faithfully balanced and n-selforthogonal bimodule for all n, then ${}_R\omega$ is just a generalized tilting module with $S = \operatorname{End}({}_R\omega)$ in the sense of Wakamatsu [4, 5]. In this case, ω_S is also a generalized tilting module with $R = \operatorname{End}(\omega_S)$ by [5, Corollary 3.2].

Definition 1.2 *Let $\mathscr{C}$ be a full subcategory of $\operatorname{mod} R$ and $A, B \in \operatorname{mod} R$. For a positive integer n, A is called an n-$\mathscr{C}$-syzygy of B, denoted by $A = \Omega^n_{\mathscr{C}}(B)$, if there exists an exact sequence $0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0 \to B \to 0$ in $\operatorname{mod} R$ with all $C_i \in \mathscr{C}$. We use $\Omega^n_{\mathscr{C}}(\operatorname{mod} R)$ to denote the full subcategory of $\operatorname{mod} R$ consisting of n-$\mathscr{C}$-syzygy modules. In case $\mathscr{C} = \operatorname{add}_R \omega$ for some module ${}_R\omega \in \operatorname{mod} R$, we call an n-$\mathscr{C}$-syzygy module n-ω-syzygy, and denote by $\Omega^n_{\omega}(\operatorname{mod} R)$ the full subcategory of $\operatorname{mod} R$ consisting of n-ω-syzygy modules.*

Let R be a left Noetherian ring, S a right Noetherian ring and ${}_R\omega_S$ a faithfully balanced and n-selforthogonal bimodule. Assume that $M \in \operatorname{mod} R$ and $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a projective presentation of M in $\operatorname{mod} R$. Then we have an exact sequence $0 \to M^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to \operatorname{Tr}_{\omega} M \to 0$ in $\operatorname{mod} S^{\mathrm{op}}$, where $\operatorname{Tr}_{\omega} M = \operatorname{Coker} f^{\omega}$. Recall from [6] that M is called *n-ω-torsionfree* if $\operatorname{Ext}^i_{S^{\mathrm{op}}}(\operatorname{Tr}_{\omega} M, \omega) = 0$ for any $1 \leq i \leq n$. Though $\operatorname{Tr}_{\omega} M$ depends on the choice of the projective presentation of M, the notion of n-ω-torsionfree modules is well defined by [6, Proposition 3]. We denote by $\mathcal{T}^n_{\omega}(\operatorname{mod} R) = \{M \in \operatorname{mod} R \text{ is } n\text{-}\omega\text{-torsionfree}\}$. In general, $\mathcal{T}^n_{\omega}(\operatorname{mod} R) \subseteq \Omega^n_{\omega}(\operatorname{mod} R)$ for any $n \geq 1$ (see [7]). We remark that when ${}_R\omega_S = {}_RR_R$, the notions of n-ω-syzygy modules and n-ω-torsionfree modules are just that of n-syzygy modules and n-torsionfree modules in [1].

Let $M \in \operatorname{mod} R$ and $n \geq 0$. We define the *ω-dimension* of M, denoted by $\omega\text{-}\dim_R M$, as $\inf\{n \mid \text{there exists an exact sequence } 0 \to \omega_n \to \cdots \to \omega_1 \to \omega_0 \to M \to 0 \text{ in } \operatorname{mod} R \text{ with all } \omega_i \in \operatorname{add}_R \omega\}$. For a full subcategory $\mathscr{C}$ of $\operatorname{mod} R$, we denote by $\operatorname{gen}^n(\mathscr{C}) = \{M \mid \text{there exists an exact sequence } C_n \to \cdots \to C_1 \to C_0 \to M \to 0 \text{ in } \operatorname{mod} R \text{ with all } C_i \in \mathscr{C}\}$. In addition, we denote by ${}^{\perp_n}_{R}\omega = \{M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, \omega) = 0 \text{ for any } 1 \leq i \leq n\}$.

In Section 3, we will prove the following

Theorem 1.3 *Let R be a left Noetherian ring, S a right Noetherian ring and ${}_R\omega_S$ a faithfully balanced and n-selforthogonal bimodule, and let $\mathscr{C}$ be a full subcategory of ${}^{\perp_n}_{R}\omega \cap \mathcal{T}^n_{\omega}(\operatorname{mod} R)$ and $M \in \operatorname{gen}^n(\mathscr{C})$ with $n \geq 1$. Then the following statements are equivalent.*

(1) *$\Omega^n_{\mathscr{C}}(M)$ is n-ω-torsionfree.*

(2) *There exists an exact sequence $0 \to B \to A \to M \to 0$ in $\operatorname{mod} R$ with $A \in {}^{\perp_n}_{R}\omega$ and*

$\omega\text{-}\dim_R B \leq n-1$.

The special case for $\mathscr{C} = \operatorname{add}_R R$ in Theorem 1.3 is a non-commutative version of Theorem 1.1. In addition, we establish the relation between the injective dimension of ${}_R\omega$ and the so-called finitistic dimension of R with respect to ω (see Section 3 for the definition) when the condition (2) in Theorem 1.3 is satisfied for any module in $\operatorname{mod} R$ (Proposition 3.7).

Let $A \in \operatorname{mod} R$ (resp. $\operatorname{mod} S^{\mathrm{op}}$) and i be a non-negative integer. Recall from [7] that the *grade* of A with respect to ω, written $\operatorname{grade}_\omega A$, is at least i if $\operatorname{Ext}_R^j(A,\omega) = 0$ (resp. $\operatorname{Ext}_{S^{\mathrm{op}}}^j(A,\omega) = 0$) for any $0 \leq j < i$. Auslander and Bridger obtained in [1] an equivalent characterization for the condition that an i-syzygy module is i-torsionfree for any $1 \leq i \leq n$ in terms of the grade of modules as follows.

Theorem 1.4 ([1, Proposition 2.26]) *Let R be a left and right Noetherian ring and $n \geq 1$. Then the following statements are equivalent.*

(1) $\Omega_R^i(M)$ *is i-torsionfree for any $M \in \operatorname{mod} R$ and $1 \leq i \leq n$.*

(2) $\operatorname{grade}_R \operatorname{Ext}_R^{i+1}(M,R) \geq i$ *for any $M \in \operatorname{mod} R$ and $1 \leq i \leq n-1$.*

Takahashi got in [3, Proposition 4.2] a C-version of Theorem 1.4 for an n-semidualizing module C over a commutative Noetherian ring. In Section 4, we will prove the following

Theorem 1.5 *Let R be a left Noetherian ring, S a right Noetherian ring and ${}_R\omega_S$ a faithfully balanced and n-selforthogonal bimodule, and let $\mathscr{C}$ be a full subcategory of ${}_R^{\perp_n}\omega \cap \mathcal{T}_\omega^n(\operatorname{mod} R)$ and $M \in \operatorname{gen}^n(\mathscr{C})$ with $n \geq 1$. Then the following statements are equivalent.*

(1) $\Omega_{\mathscr{C}}^i(M)$ *is i-ω-torsionfree for any $1 \leq i \leq n$.*

(2) $\operatorname{grade}_\omega \operatorname{Ext}_R^{i+1}(M,\omega) \geq i$ *for any $1 \leq i \leq n-1$.*

The special case for $\mathscr{C} = \operatorname{add}_R R$ in Theorem 1.5 is a non-commutative version of [3, Proposition 4.2]. Putting ${}_R\omega_S = {}_RR_R$ and $\mathscr{C} = \operatorname{add}_R R$ in Theorem 1.5, we get Theorem 1.4.

2 Preliminaries

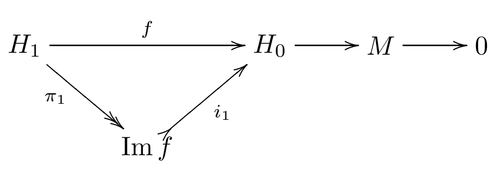
Let R and S be rings and ${}_R\omega_S$ an (R,S)-bimodule, and let $M \in \operatorname{mod} R$ and $\sigma_M : M \to M^{\omega\omega}$ via $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^\omega$ be the canonical evaluation homomorphism. Recall that $M \in \operatorname{mod} R$ is called *ω-torsionless* if σ_M is a monomorphism; and M is called *ω-reflexive* if σ_M is an isomorphism.

Lemma 2.1 *Let R and S be rings and ${}_R\omega_S$ an (R,S)-bimodule, and let $M \in \operatorname{mod} R$ and $H_1 \xrightarrow{f} H_0 \to M \to 0$ be an exact sequence in $\operatorname{mod} R$ with H_0, H_1 ω-reflexive. Put $X = \operatorname{Coker} f^\omega$. Then we have*

(1) *If* $\operatorname{Ext}_{S^{\mathrm{op}}}^1(H_0^\omega,\omega) = 0 = \operatorname{Ext}_{S^{\mathrm{op}}}^1(H_1^\omega,\omega)$*, then* $\operatorname{Ker}\sigma_M \cong \operatorname{Ext}_{S^{\mathrm{op}}}^1(X,\omega)$*; if further* $\operatorname{Ext}_{S^{\mathrm{op}}}^2(H_1^\omega,\omega) = 0$*, then* $\operatorname{Coker}\sigma_M \cong \operatorname{Ext}_{S^{\mathrm{op}}}^2(X,\omega)$.

(2) *If* $\operatorname{Ext}_R^1(H_0,\omega) = 0 = \operatorname{Ext}_R^1(H_1,\omega)$*, then* $\operatorname{Ker}\sigma_X \cong \operatorname{Ext}_R^1(M,\omega)$*; if further* $\operatorname{Ext}_R^2(H_0,\omega) = 0$*, then* $\operatorname{Coker}\sigma_X \cong \operatorname{Ext}_R^2(M,\omega)$.

Proof (1) Let



be an exact sequence in $\operatorname{mod} R$ with H_0, H_1 ω-reflexive, $\operatorname{Ext}^1_{S^{\rm op}}(H_0^\omega, \omega) = 0 = \operatorname{Ext}^1_{S^{\rm op}}(H_1^\omega, \omega)$ and $f = i_1\pi_1$ the natural epic-monic decomposition of f. Then we get an exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^\omega & \longrightarrow & H_0^\omega & \xrightarrow{\quad f^\omega \quad} & H_1^\omega & \longrightarrow X \longrightarrow & 0 \\
 & & & & {\scriptstyle \pi_2}\searrow & & \nearrow{\scriptstyle i_2} & & \\
 & & & & & \operatorname{Im} f^\omega & & &
\end{array}$$

in $\operatorname{mod} S^{\rm op}$ with $f^\omega = i_2\pi_2$ the natural epic-monic decomposition of f^ω, and the following exact commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Im} f & \xrightarrow{i_1} & H_0 & \longrightarrow & M & \longrightarrow & & & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \sigma_{H_0}} & & \downarrow{\scriptstyle \sigma_M} & & & & \\
0 & \longrightarrow & (\operatorname{Im} f^\omega)^\omega & \xrightarrow{\pi_2^\omega} & H_0^{\omega\omega} & \longrightarrow & M^{\omega\omega} & \longrightarrow & \operatorname{Ext}^1_{S^{\rm op}}(\operatorname{Im} f^\omega, \omega) & \longrightarrow & 0,
\end{array}$$

where σ_{H_0} is an isomorphism and g is an induced homomorphism. By the snake lemma, we have that g is monic and $\operatorname{Ker} \sigma_M \cong \operatorname{Coker} g$.

Because $\sigma_{H_0} i_1 = \pi_2^\omega g$, $(\sigma_{H_0} i_1)\pi_1 = (\pi_2^\omega g)\pi_1$ and so $\sigma_{H_0} f = \pi_2^\omega g \pi_1$. Since $\sigma_{H_0} f = f^{\omega\omega}\sigma_{H_1}$ and $f^{\omega\omega} = \pi_2^\omega i_2^\omega$, $\pi_2^\omega i_2^\omega \sigma_{H_1} = \pi_2^\omega g\pi_1$. Since π_2^ω is monic, $i_2^\omega \sigma_{H_1} = g\pi_1$. Thus $\operatorname{Im}(i_2^\omega \sigma_{H_1}) \subseteq \operatorname{Im} g$, and therefore, by [8, Theorem 3.6], there exists an induced homomorphism h such that the following diagram is commutative and exact:

$$\begin{array}{ccccccccc}
 & & H_1 & \xrightarrow{i_2^\omega \sigma_{H_1}} & (\operatorname{Im} f^\omega)^\omega & \longrightarrow & \operatorname{Ext}^1_{S^{\rm op}}(X, \omega) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \| & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \operatorname{Im} f & \xrightarrow{g} & (\operatorname{Im} f^\omega)^\omega & \longrightarrow & \operatorname{Coker} g & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

By the snake lemma, h is an isomorphism, so $\operatorname{Ker} \sigma_M \cong \operatorname{Coker}\ g \cong \operatorname{Ext}^1_{S^{\rm op}}(X, \omega)$.

If further $\operatorname{Ext}^2_{S^{\rm op}}(H_1^\omega, \omega) = 0$, then $\operatorname{Ext}^1_{S^{\rm op}}(\operatorname{Im} f^\omega, \omega) \cong \operatorname{Ext}^2_{S^{\rm op}}(X, \omega)$. So we have $\operatorname{Coker} \sigma_M \cong \operatorname{Ext}^1_{S^{\rm op}}(\operatorname{Im} f^\omega, \omega) (\cong \operatorname{Ext}^2_{S^{\rm op}}(X, \omega))$.

(2) From the exact sequence $H_1 \xrightarrow{f} H_0 \to M \to 0$ in $\operatorname{mod} R$, we get an exact sequence $0 \to M^\omega \to H_0^\omega \xrightarrow{f^\omega} H_1^\omega \to X \to 0$ in $\operatorname{mod} S^{\rm op}$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & H_1 & \xrightarrow{f} & H_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_{H_1}} & & \downarrow{\scriptstyle \sigma_{H_0}} & & \downarrow & & \\
0 \longrightarrow & X^\omega & \longrightarrow H_1^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & H_0^{\omega\omega} & \longrightarrow & \operatorname{Coker} f^{\omega\omega} & \longrightarrow & 0.
\end{array}$$

Because both σ_{H_0} and σ_{H_1} are isomorphisms, $M \cong \operatorname{Coker} f^{\omega\omega}$. Notice that both H_0^ω and H_1^ω are also ω-reflexive, then it is not difficult to see that the proof of (2) is analogous to that of (1). So we omit it. $\square$

From now on, R is a left Noetherian ring, S is a right Noetherian ring and ${}_R\omega_S$ is a faithfully balanced and n-selforthogonal bimodule with $n \geq 1$.

By Lemma 2.1, a module $M \in \operatorname{mod} R$ is ω-torsionless if and only if it is 1-ω-torsionfree, and M is ω-reflexive if and only if it is 2-ω-torsionfree (in this case $n \geq 2$). In addition, it is easy to see that any projective module in $\operatorname{mod} R$ (resp. $\operatorname{mod} S^{\mathrm{op}}$) and any module in $\operatorname{add}_R \omega$ (resp. $\operatorname{add} \omega_S$) are ω-reflexive.

Lemma 2.2 ([7, Lemma 2.9]) *Let $n \geq 3$. Then an ω-reflexive module M in $\operatorname{mod} R$ is n-ω-torsionfree if and only if $\operatorname{Ext}^i_{S^{\mathrm{op}}}(M^\omega, \omega) = 0$ for any $1 \leq i \leq n-2$.*

Lemma 2.3 *Let $\mathscr{C}$ be a full subcategory of ${}_R^{\perp_2}\omega \cap \mathcal{T}^2_\omega(\operatorname{mod} R)$ and $M \in \operatorname{gen}^2(\mathscr{C})$. Then the following statements are equivalent.*

(1) *$\Omega^2_{\mathscr{C}}(M)$ is ω-reflexive.*

(2) *$[\operatorname{Ext}^2_R(M, \omega)]^\omega = 0$.*

Proof Let $M \in \operatorname{gen}^2(\mathscr{C})$ and

$$0 \to \Omega^2_{\mathscr{C}}(M) \xrightarrow{f} C \to \Omega^1_{\mathscr{C}}(M) \to 0$$

be an exact sequence in $\operatorname{mod} R$ with $C \in \mathscr{C}$. By applying the functor $(-)^\omega$, we get the following exact sequence:

$$0 \to [\Omega^1_{\mathscr{C}}(M)]^\omega \to C^\omega \xrightarrow{f^\omega} [\Omega^2_{\mathscr{C}}(M)]^\omega \to \operatorname{Ext}^2_R(M, \omega) \to 0. \tag{2.1}$$

(1) $\Rightarrow$ (2) We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \Omega^2_{\mathscr{C}}(M) & \longrightarrow & C \\
 & & & & \Big\downarrow {\scriptstyle \sigma_{\Omega^2_{\mathscr{C}}(M)}} & & \Big\downarrow {\scriptstyle \sigma_C} \\
0 & \longrightarrow & [\operatorname{Ext}^2_R(M,\omega)]^\omega & \longrightarrow & [\Omega^2_{\mathscr{C}}(M)]^{\omega\omega} & \longrightarrow & C^{\omega\omega}
\end{array}$$

with σ_C an isomorphism. By (1), $\Omega^2_{\mathscr{C}}(M)$ is ω-reflexive, so $\sigma_{\Omega^2_{\mathscr{C}}(M)}$ is also an isomorphism and hence $[\operatorname{Ext}^2_R(M, \omega)]^\omega = 0$.

(2) $\Rightarrow$ (1) Set $N = \operatorname{Im} f^\omega$ and decompose (2.1) into two short exact sequences: $0 \to [\Omega^1_{\mathscr{C}}(M)]^\omega \to C^\omega \xrightarrow{\pi} N \to 0$ and $0 \to N \xrightarrow{\alpha} [\Omega^2_{\mathscr{C}}(M)]^\omega \to \operatorname{Ext}^2_R(M, \omega) \to 0$. By applying $(-)^\omega$ to the former exact sequence, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^2_{\mathscr{C}}(M) & \xrightarrow{f} & C & \longrightarrow & \Omega^1_{\mathscr{C}}(M) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \beta} & & \Big\downarrow {\scriptstyle \sigma_C} & & \Big\downarrow {\scriptstyle \sigma_{\Omega^1_{\mathscr{C}}(M)}} & & \\
0 & \longrightarrow & N^\omega & \xrightarrow{\pi^\omega} & C^{\omega\omega} & \longrightarrow & [\Omega^1_{\mathscr{C}}(M)]^{\omega\omega} & &
\end{array}$$

with σ_C an isomorphism and $\sigma_{\Omega^1_{\mathscr{C}}(M)}$ a monomorphism. So β is an isomorphism. Because $\pi^\omega \beta = \sigma_C f = f^{\omega\omega} \sigma_{\Omega^2_{\mathscr{C}}(M)} = \pi^\omega \alpha^\omega \sigma_{\Omega^2_{\mathscr{C}}(M)}$ and π^ω is a monomorphism, $\beta = \alpha^\omega \sigma_{\Omega^2_{\mathscr{C}}(M)}$ and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & & \Omega^2_{\mathscr{C}}(M) & =\!= & \Omega^2_{\mathscr{C}}(M) \\
 & & & & \Big\downarrow {\scriptstyle \sigma_{\Omega^2_{\mathscr{C}}(M)}} & & \Big\downarrow {\scriptstyle \beta} \\
0 & \longrightarrow & [\operatorname{Ext}^2_R(M,\omega)]^\omega & \longrightarrow & [\Omega^2_{\mathscr{C}}(M)]^{\omega\omega} & \xrightarrow{\alpha^\omega} & N^\omega.
\end{array}$$

Then $\operatorname{Coker} \sigma_{\Omega^2_{\mathscr{C}}(M)} \cong [\operatorname{Ext}^2_R(M, \omega)]^\omega = 0$ by the snake lemma and (2). Because $\sigma_{\Omega^2_{\mathscr{C}}(M)}$ is a monomorphism, $\sigma_{\Omega^2_{\mathscr{C}}(M)}$ is an isomorphism and $\Omega^2_{\mathscr{C}}(M)$ is ω-reflexive. $\square$

Let $\mathscr{C} \supseteq \mathscr{D}$ be full subcategories of $\operatorname{mod} R$ and $C \in \mathscr{C}, D \in \mathscr{D}$. The homomorphism $C \to D$ is said to be a *left $\mathscr{D}$-approximation* of C if $\operatorname{Hom}_R(D, X) \to \operatorname{Hom}_R(C, X)$ is epic for any $X \in \mathscr{D}$. Dually, the notion of *right $\mathscr{D}$-approximations* is defined (see [9]).

Lemma 2.4 ([6, Theorem 1]) *The following statements are equivalent for any $M \in \operatorname{mod} R$.*

(1) *M is n-ω-torsionfree.*

(2) *There exists an exact sequence $0 \to M \xrightarrow{f_1} \omega_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} \omega_n$ in $\operatorname{mod} R$ such that $\operatorname{Im} f_i \to \omega_i$ is a left $\operatorname{add}_R \omega$-approximation of $\operatorname{Im} f_i$ for any $1 \leq i \leq n$.*

In particular, under the condition (2), we have $\operatorname{Ext}^1_R(\operatorname{Coker} f_i, \omega) = 0$ for any $1 \leq i \leq n$.

3 When n-$\mathscr{C}$-syzygy Modules Are n-ω-torsionfree

Before proving the main result in this section, we need some lemmas.

Lemma 3.1 *Let $\mathscr{C}$ be a full subcategory of $T^n_\omega(\operatorname{mod} R)$ and $M \in \operatorname{mod} R$ n-ω-torsionfree. If there exists an exact sequence $0 \to M \to C_0 \to N \to 0$ in $\operatorname{mod} R$ with $C_0 \in \mathscr{C}$ with $\operatorname{Ext}^1_R(N, \omega) = 0$, then N is $(n-1)$-ω-torsionfree.*

Proof The case for $n = 1$ is trivial. Now suppose $n \geq 2$. Since $\operatorname{Ext}^1_R(N, \omega) = 0$, by applying $(-)^\omega$ to the exact sequence $0 \to M \to C_0 \to N \to 0$, we get an exact sequence $0 \to N^\omega \to C_0^\omega \to M^\omega \to 0$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_M} & & \downarrow{\scriptstyle \sigma_{C_0}} & & \downarrow{\scriptstyle \sigma_N} & & \\
0 & \longrightarrow & M^{\omega\omega} & \longrightarrow & C_0^{\omega\omega} & \xrightarrow{\alpha} & N^{\omega\omega}. & &
\end{array}$$

Because both σ_M and σ_{C_0} are isomorphisms, σ_N is monic and N is ω-torsionless. When $n \geq 3$, we have $\operatorname{Ext}^i_{S^{\mathrm{op}}}(M^\omega, \omega) = 0$ for any $1 \leq i \leq n-2$, so α is epic, which implies that σ_N is an isomorphism and N is ω-reflexive. Since $C_0 \in \mathscr{C}$, $\operatorname{Ext}^i_{S^{\mathrm{op}}}(C_0^\omega, \omega) = 0$ for any $1 \leq i \leq n-2$. Thus $\operatorname{Ext}^i_{S^{\mathrm{op}}}(N^\omega, \omega) = 0$ for $1 \leq i \leq n-3$ (when $n \geq 4$) and N is $(n-1)$-ω-torsionfree by Lemma 2.2. □

Lemma 3.2 *Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\operatorname{mod} R$ satisfying $\operatorname{Ext}^1_R(C, \omega) = 0$. If both A and C are n-ω-torsionfree, then so is B.*

Proof Let $P_1 \to P_0 \to A \to 0$ and $Q_1 \to Q_0 \to C \to 0$ be projective presentations of A and C in $\operatorname{mod} R$ respectively. Then we have the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Because $\mathrm{Ext}_R^1(C,\omega)=0$, by applying $(-)^{\omega}$ to the above diagram, we get the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{\omega} & \longrightarrow & B^{\omega} & \longrightarrow & A^{\omega} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_0^{\omega} & \longrightarrow & P_0^{\omega}\oplus Q_0^{\omega} & \longrightarrow & P_0^{\omega} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1^{\omega} & \longrightarrow & P_1^{\omega}\oplus Q_1^{\omega} & \longrightarrow & P_1^{\omega} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Tr}_{\omega}C & & \mathrm{Tr}_{\omega}B & & \mathrm{Tr}_{\omega}A & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Then by the snake lemma, $0\to \mathrm{Tr}_{\omega}C\to \mathrm{Tr}_{\omega}B\to \mathrm{Tr}_{\omega}A\to 0$ is exact. Now the assertion follows easily. □

The following is one of the main results in this paper.

Theorem 3.3 *Let $\mathscr{C}$ be a full subcategory of ${}_R^{\perp n}\omega\cap\mathcal{T}_{\omega}^n(\mathrm{mod}\,R)$ and $M\in\mathrm{gen}^n(\mathscr{C})$. Then the following statements are equivalent.*

(1) *$\Omega_{\mathscr{C}}^n(M)$ is n-ω-torsionfree.*

(2) *There exists an exact sequence $0\to B\to A\to M\to 0$ in $\mathrm{mod}\,R$ such that $A\in{}_R^{\perp n}\omega$ and ω-$\mathrm{dim}_R\,B\le n-1$.*

Proof Let $M\in\mathrm{gen}^n(\mathscr{C})$. Then there exists an exact sequence $0\to\Omega_{\mathscr{C}}^n(M)\to C_{n-1}\to C_{n-2}\to\cdots\to C_1\to C_0\to M\to 0$ in $\mathrm{mod}\,R$ with each $C_i\in\mathscr{C}$.

(1) $\Rightarrow$ (2) Because $\Omega_{\mathscr{C}}^n(M)$ is n-ω-torsionfree by assumption, there exists an exact sequence $0\to\Omega_{\mathscr{C}}^n(M)\to\omega_0\to N_1\to 0$ in $\mathrm{mod}\,R$ with $\omega_0\in\mathrm{add}_R\,\omega$ and $\mathrm{Ext}_R^1(N_1,\omega)=0$ by Lemma 2.4. Consider the following push-out diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_{\mathscr{C}}^n(M) & \longrightarrow & \omega_0 & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C_{n-1} & \longrightarrow & A_1 & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_{\mathscr{C}}^{n-1}(M) & = & \Omega_{\mathscr{C}}^{n-1}(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Then $\mathrm{Ext}_R^1(A_1,\omega)=0$. If $n=1$, then the middle column in the above diagram is the desired exact sequence.

Now suppose $n\geq 2$. Because N_1 is $(n-1)$-ω-torsionfree by Lemma 3.1, A_1 is $(n-1)$-ω-torsionfree by Lemma 3.2 and the exactness of the middle row in the above diagram. Then by Lemma 2.4, there exists an exact sequence

$$0\to A_1\to\omega_1\to N_2\to 0$$

in $\operatorname{mod} R$ with $\omega_1\in\operatorname{add}_R\omega$ and $\mathrm{Ext}_R^1(N_2,\omega)=0$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \omega_0 & = & \omega_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A_1 & \to & \omega_1 & \to & N_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \Omega_{\mathscr{C}}^{n-1}(M) & \to & B_2 & \to & N_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Then

$$\omega\text{-}\dim_R B_2\leq 1$$

and

$$\mathrm{Ext}_R^2(N_2,\omega)=0\,(=\mathrm{Ext}_R^1(N_1,\omega)).$$

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega_{\mathscr{C}}^{n-1}(M) & \to & B_2 & \to & N_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C_{n-2} & \to & A_2 & \to & N_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_{\mathscr{C}}^{n-2}(M) & = & \Omega_{\mathscr{C}}^{n-2}(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

From the middle row in the above diagram, we know that

$$\mathrm{Ext}_R^i(A_2,\omega)=0,\quad \text{for } i=1,2.$$

Thus, if $n=2$, then the middle column in this diagram is the desired exact sequence.

If $n \geq 3$, then we get similarly that both N_2 and A_2 are $(n-2)$-ω-torsionfree. So there exists an exact sequence $0 \to A_2 \to \omega_2 \to N_3 \to 0$ in $\operatorname{mod} R$ with $\omega_2 \in \operatorname{add}_R \omega$ and $\operatorname{Ext}^1_R(N_3, \omega) = 0$ by Lemma 2.4. Similarly to the above, we construct the following two push-out diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & B_2 & = & B_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A_2 & \to & \omega_2 & \to & N_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \Omega^{n-2}_{\mathscr{C}}(M) & \to & B_3 & \to & N_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega^{n-2}_{\mathscr{C}}(M) & \to & B_3 & \to & N_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C_{n-3} & \to & A_3 & \to & N_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{n-3}_{\mathscr{C}}(M) & = & \Omega^{n-3}_{\mathscr{C}}(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

If $n = 3$, then the middle column in the above diagram is the desired exact sequence. If $n \geq 4$, then, iterating this procedure, we get an exact sequence

$$0 \to B_n \to A_n \to M \to 0$$

in $\operatorname{mod} R$ with $A_n \in {}_R^{\perp_n}\omega$ and

$$\omega\text{-}\dim_R B_n \leq n - 1.$$

(2) $\Rightarrow$ (1) Let $0 \to B \to A \to M \to 0$ be an exact sequence in $\operatorname{mod} R$ with $A \in {}_R^{\perp_n}\omega$ and ω-$\dim_R B \leq n-1$. Then we have an exact sequence

$$0 \to \omega_{n-1} \xrightarrow{f_{n-1}} \omega_{n-2} \xrightarrow{f_{n-2}} \cdots \to \omega_0 \xrightarrow{f_0} B \to 0$$

in $\operatorname{mod} R$ with each $\omega_i \in \operatorname{add}_R \omega$. Put

$$B_i = \operatorname{Im} f_i$$

for each i (note: $B_0 = B$ and $B_{n-1} = \omega_{n-1}$). Consider the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^1_{\mathscr{C}}(M) & = & \Omega^1_{\mathscr{C}}(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & L & \to & C_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & A & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

It is easy to see that $\mathrm{Ext}^1_R(C_0, B) = 0$, so the middle row in the above diagram splits and $L \cong B \oplus C_0$. Adding C_0 to the exact sequence $0 \to B_1 \to \omega_0 \to B \to 0$, we get an exact sequence $0 \to B_1 \to \omega_0 \oplus C_0 \to B \oplus C_0 \to 0$.

Consider the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & B_1 & = & B_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A_1 & \to & \omega_0 \oplus C_0 & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \Omega^1_{\mathscr{C}}(M) & \to & B \oplus C_0 & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Applying a similar argument to the first column in the above diagram, we get exact sequences $0 \to A_{i+1} \to \omega_i \oplus C_i \to A_i \to 0$ for any $0 \le i \le n-1$, where $A_0 = A$ and $A_n = \Omega^n_{\mathscr{C}}(M)$. The assumption yields that $\mathrm{Ext}^i_R(A_0, \omega) = 0 = \mathrm{Ext}^i_R(\omega_0 \oplus C_0, \omega)$ for any $1 \le i \le n$. So we get an exact sequence $0 \to A_0^{\omega} \to (\omega_0 \oplus C_0)^{\omega} \to A_1^{\omega} \to 0$ and $\mathrm{Ext}^i_R(A_1, \omega) = 0$ for any $1 \le i \le n-1$. Inductively, we get an exact sequence $0 \to A_i^{\omega} \to (\omega_i \oplus C_i)^{\omega} \to A_{i+1}^{\omega} \to 0$ for any $0 \le i \le n-1$ and $\mathrm{Ext}^j_R(A_i, \omega) = 0$ for any $1 \le j \le n-i$.

Because $\omega_0 \oplus C_0$ is ω-torsionless, A_1 is also ω-torsionless. If $n \ge 2$, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & A_2 & \to & \omega_1 \oplus C_1 & \to & A_1 & \to & 0 \\
 & & \downarrow{\scriptstyle \sigma_{A_2}} & & \downarrow{\scriptstyle \sigma_{\omega_1 \oplus C_1}} & & \downarrow{\scriptstyle \sigma_{A_1}} & & \\
0 & \to & A_2^{\omega\omega} & \to & (\omega_1 \oplus C_1)^{\omega\omega} & \to & A_1^{\omega\omega}. & &
\end{array}
$$

Then σ_{A_2} is an isomorphism by the snake lemma and so A_2 is ω-reflexive. If $n \geq 3$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_3 & \longrightarrow & \omega_2 \oplus C_2 & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_{A_3}} & & \downarrow{\scriptstyle \sigma_{\omega_2 \oplus C_2}} & & \downarrow{\scriptstyle \sigma_{A_2}} & & \\ 0 & \longrightarrow & A_3^{\omega\omega} & \longrightarrow & (\omega_2 \oplus C_2)^{\omega\omega} & \longrightarrow & A_2^{\omega\omega} & \longrightarrow & \mathrm{Ext}^1_{S^{\mathrm{op}}}(A_3^{\omega}, \omega) \longrightarrow 0. \end{array}$$

It follows that σ_{A_3} is an isomorphism and $\mathrm{Ext}^1_{S^{\mathrm{op}}}(A_3^{\omega}, \omega) = 0$, which implies that A_3 is 3-ω-torsionfree by Lemma 2.2. Repeating this procedure, we get that A_i is i-ω-torsionfree for any $1 \leq i \leq n$. In particular, $\Omega^n_{\mathscr{C}}(M)(= A_n)$ is n-ω-torsionfree. □

As an immediate consequence of Theorem 3.3, we get the following

Corollary 3.4 *Let* $\mathscr{C}$ *be a full subcategory of* ${}^{\perp_n}_{R}\omega \cap \mathcal{T}^n_{\omega}(\mathrm{mod}\, R)$ *containing* $\mathrm{add}_R R$*. Then the following statements are equivalent for a module* $M \in \mathrm{mod}\, R$*.*

(1) $\Omega^n_{\mathscr{C}}(M)$ *is* n*-*ω*-torsionfree.*

(2) *There exists an exact sequence* $0 \to B \to A \to M \to 0$ *in* $\mathrm{mod}\, R$ *such that* $A \in {}^{\perp_n}_{R}\omega$ *and* $\omega\text{-}\dim_R B \leq n-1$*.*

It is easy to see that the exact sequence in Corollary 3.4 (2) is a right ${}^{\perp_n}_{R}\omega$-approximation of M.

Recall from [10] that a module M in $\mathrm{mod}\, R$ is said to have *generalized Gorenstein dimension zero* with respect to ω if the following conditions are satisfied: (1) M is ω-reflexive; and (2) $\mathrm{Ext}^i_R(M, \omega) = 0 = \mathrm{Ext}^i_{S^{\mathrm{op}}}(M^{\omega}, \omega)$ for any $i \geq 1$. We use $\mathcal{G}_{\omega}$ to denote the full subcategory of $\mathrm{mod}\, R$ consisting of the modules with generalized Gorenstein dimension zero with respect to ω. In addition, we denote by $\overline{\mathrm{add}\,_R\omega} = \mathrm{add}\,_R\omega \cup \mathrm{add}\,_R R$.

In the following result, the equivalence between (1) and (4) is a non-commutative version of Theorem 1.1 due to Takahashi.

Corollary 3.5 *The following statements are equivalent for a module* $M \in \mathrm{mod}\, R$*.*

(1) $\Omega^n_R(M)$ *is* n*-*ω*-torsionfree.*

(2) $\Omega^n_{\mathcal{G}_{\omega}}(M)$ *is* n*-*ω*-torsionfree.*

(3) $\Omega^n_{\overline{\mathrm{add}\,_R\omega}}(M)$ *is* n*-*ω*-torsionfree.*

(4) *There exists an exact sequence* $0 \to B \to A \to M \to 0$ *in* $\mathrm{mod}\, R$ *with* $A \in {}^{\perp_n}_{R}\omega$ *and* $\omega\text{-}\dim_R B \leq n-1$*.*

Proof We get (1) ⇔ (4), (2) ⇔ (4) and (3) ⇔ (4) by putting $\mathscr{C} = \mathrm{add}\,_R R$, $\mathscr{C} = \mathcal{G}_{\omega}$ and $\mathscr{C} = \overline{\mathrm{add}\,_R\omega}$ in Corollary 3.4, respectively. □

The equivalence between (1) and (2) in the following result is an ω-version of a result of Auslander and Bridger in [1, Proposition 2.21].

Corollary 3.6 *The following statements are equivalent for a module* $M \in \mathrm{gen}^n(\mathrm{add}\,_R\omega)$*.*

(1) $\Omega^n_{\omega}(M)$ *is* n*-*ω*-torsionfree.*

(2) *There exists an exact sequence* $0 \to B \to A \to M \to 0$ *in* $\mathrm{mod}\, R$ *with* $A \in {}^{\perp_n}_{R}\omega$ *and* $\omega\text{-}\dim_R B \leq n-1$*.*

(3) *There exists an exact sequence* $0 \to \Omega^1_{\omega}(M) \to B' \to A' \to 0$ *in* $\mathrm{mod}\, R$ *with* $A' \in {}^{\perp_n}_{R}\omega$ *and* $\omega\text{-}\dim_R B' \leq n-1$*.*

Proof Putting $\mathscr{C} = \mathrm{add}\,{}_R\omega$ in Theorem 3.3, we get (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3) Let $0 \to \Omega_\omega^1(M) \to \omega_0 \to M \to 0$ be an exact sequence in $\mathrm{mod}\,R$ with $\omega_0 \in \mathrm{add}_R\,\omega$.

Assume that the condition (2) is satisfied, that is, there exists an exact sequence $0 \to B \to A \to M \to 0$ in $\mathrm{mod}\,R$ with $A \in {}_R^{\perp_n}\omega$ and ω-$\dim_R B \le n-1$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_\omega^1(M) & = & \Omega_\omega^1(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & B' & \to & \omega_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & A & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Note that $\mathrm{add}_R\,\omega \subseteq {}_R^{\perp_n}\omega$. So the middle row in the above diagram splits and $B' \cong B \oplus \omega_0$, which implies that ω-$\dim_R B' \le n-1$. Thus the middle column is the desired exact sequence in (3).

Conversely, assume that the condition (3) is satisfied, that is, there exists an exact sequence $0 \to \Omega_\omega^1(M) \to B' \to A' \to 0$ in $\mathrm{mod}\,R$ with $A' \in {}_R^{\perp_n}\omega$ and ω-$\dim_R B' \le n-1$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega_\omega^1(M) & \to & B' & \to & A' & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \omega_0 & \to & A & \to & A' & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

From the middle row in the above diagram, we get that $A \in {}_R^{\perp_n}\omega$. So the middle column is the desired exact sequence in (2). $\square$

We define ω-fin.dim $R = \sup\{\omega\text{-}\dim_R M \mid M \in \mathrm{mod}\,R$ with $\omega\text{-}\dim_R M < \infty\}$, and denote the injective dimension of ${}_R\omega$ by $\mathrm{id}_R\,\omega$. In the following result we establish the relation between $\mathrm{id}_R\,\omega$ and ω-fin.dim R when the condition (4) in Corollary 3.5 (that is, Theorem 3.3 (2)) is satisfied for any module in $\mathrm{mod}\,R$.

Proposition 3.7 (1) *If* $\mathrm{id}_R\,\omega \le n$*, then* ω*-fin.dim* $R \le n$*.*

(2) *Assume that the condition* (4) *in Corollary* 3.5 *is satisfied for any module* M *in* $\mathrm{mod}\,R$*. If* ω*-fin.dim* $R \le n-2$*, then* $\mathrm{id}_R\,\omega \le n-1$*.*

(3) *Let* ${}_R\omega$ *be generalized tilting with* $S = \mathrm{End}({}_R\omega)$*. If the condition* (4) *in Corollary* 3.5 *is satisfied for any module* M *in* $\mathrm{mod}\,R$ *and any positive integer* n*, then* ω*-fin.dim* $R \le \mathrm{id}_R\,\omega \le \omega$*-fin.dim* $R+1$*.*

Proof (1) Assume that $\mathrm{id}_R\,\omega \le n$ and $M \in \mathrm{mod}\,R$ with $\omega\text{-}\dim_R M = m(<\infty)$. Then there exists an exact sequence $0 \to \omega_m \to \cdots \to \omega_1 \to \omega_0 \to M \to 0$ in $\mathrm{mod}\,R$ with all $\omega_i \in \mathrm{add}_R\,\omega$. If $m > n$, we put $K_n = \mathrm{Im}(\omega_n \to \omega_{n-1})$. Then $\mathrm{Ext}^i_R(K_n, \omega) \cong \mathrm{Ext}^{n+i}_R(M, \omega) = 0$ for any $i \ge 1$. So the exact sequence $0 \to \omega_m \to \cdots \to \omega_n \to K_n \to 0$ splits, and hence $K_n \in \mathrm{add}_R\,\omega$. Thus $\omega\text{-}\dim_R M \le n$ and ω-fin.dim $R \le n$.

(2) Let $M \in \mathrm{mod}\,R$. Then by assumption there exists an exact sequence $0 \to B \to A \to M \to 0$ in $\mathrm{mod}\,R$ with $A \in {}^{\perp_n}_{R}\omega$ and $\omega\text{-}\dim_R B \le n-1$. If ω-fin.dim $R \le n-2$, then $\omega\text{-}\dim_R B \le n-2$. So $\mathrm{Ext}^n_R(M, \omega) \cong \mathrm{Ext}^{n-1}_R(B, \omega) = 0$, and hence $\mathrm{id}_R\,\omega \le n-1$.

(3) It is an immediate consequence of (1) and (2). □

4 The Grade of Modules with Respect to ω

The following is another main result in this paper.

Theorem 4.1 *Let* $\mathscr{C}$ *be a full subcategory of* ${}^{\perp_n}_{R}\omega \cap \mathcal{T}^n_\omega(\mathrm{mod}\,R)$ *and* $M \in \mathrm{gen}^n(\mathscr{C})$*. Then the following statements are equivalent.*

(1) $\Omega^i_{\mathscr{C}}(M)$ *is* i*-*ω*-torsionfree for any* $1 \le i \le n$*.*

(2) $\mathrm{grade}_\omega\,\mathrm{Ext}^{i+1}_R(M, \omega) \ge i$ *for any* $1 \le i \le n-1$*.*

Proof We proceed by induction on n. The case for $n = 1$ is trivial, and the case for $n = 2$ follows from Lemma 2.3. Now suppose that $n \ge 3$ and $0 \to \Omega^n_{\mathscr{C}}(M) \to C_{n-1} \xrightarrow{f} C_{n-2} \to \cdots \to C_0 \to M \to 0$ is an exact sequence in $\mathrm{mod}\,R$ with all $C_i \in \mathscr{C}$. Put $N = \mathrm{Coker}\,f^\omega$.

(2) $\Rightarrow$ (1) By the induction hypothesis, $\Omega^n_{\mathscr{C}}(M)$ is $(n-1)$-ω-torsionfree (and hence ω-reflexive). We claim that $\mathrm{Ext}^i_{S^{\mathrm{op}}}(N, \omega) = 0$ for any $1 \le i \le n-2$.

By [7, Lemma 2.4], $\Omega^n_{\mathscr{C}}(M) \cong N^\omega$ and $[\Omega^n_{\mathscr{C}}(M)]^\omega \cong N^{\omega\omega}$. If $n = 3$, then $\mathrm{Coker}\,f$ $(= \Omega^{n-2}_{\mathscr{C}}(M))$ is a submodule of C_0 and so $\mathrm{Coker}\,f$ is ω-torsionless. By Lemma 2.1, $\mathrm{Ext}^1_{S^{\mathrm{op}}}(N, \omega) \cong \mathrm{Ker}\,\sigma_{\mathrm{Coker}\,f} = 0$. If $n = 4$, then $\mathrm{Coker}\,f$ is ω-reflexive by the induction hypothesis. Thus

$$\mathrm{Ext}^1_{S^{\mathrm{op}}}(N, \omega) \cong \mathrm{Ker}\,\sigma_{\mathrm{Coker}\,f} = 0 \quad \text{and} \quad \mathrm{Ext}^2_{S^{\mathrm{op}}}(N, \omega) \cong \mathrm{Coker}\,\sigma_{\mathrm{Coker}\,f} = 0$$

and the case for $n = 4$ follows. If $n \ge 5$, then $\mathrm{Coker}\,f$ is $(n-2)$-ω-torsionfree again by the induction hypothesis. Thus

$$\mathrm{Ext}^i_{S^{\mathrm{op}}}((\mathrm{Coker}\,f)^\omega, \omega) = 0$$

for any $1 \le i \le n-4$ by Lemma 2.2. It follows from the exact sequence

$$0 \to (\mathrm{Coker}\,f)^\omega \to C^\omega_{n-2} \xrightarrow{f^\omega} C^\omega_{n-1} \to N \to 0$$

that $\mathrm{Ext}^i_{S^{\mathrm{op}}}(N, \omega) = 0$ for any $3 \le i \le n-2$. So $\mathrm{Ext}^i_{S^{\mathrm{op}}}(N, \omega) = 0$ for any $1 \le i \le n-2$. The claim is proved.

By Lemma 2.1, we have an exact sequence:

$$0 \to \mathrm{Ext}_R^1(\mathrm{Coker}\, f, \omega) \to N \xrightarrow{\sigma_N} N^{\omega\omega} \to \mathrm{Ext}_R^2(\mathrm{Coker}\, f, \omega) \to 0.$$

Then

$$\mathrm{Ker}\, \sigma_N \cong \mathrm{Ext}_R^1(\mathrm{Coker}\, f, \omega) \cong \mathrm{Ext}_R^{n-1}(M, \omega)$$

and

$$\mathrm{Coker}\, \sigma_N \cong \mathrm{Ext}_R^2(\mathrm{Coker}\, f, \omega) \cong \mathrm{Ext}_R^n(M, \omega).$$

So we get the following exact sequences:

$$0 \to \mathrm{Ext}_R^{n-1}(M, \omega) \to N \xrightarrow{\pi} \mathrm{Im}\, \sigma_N \to 0, \tag{4.1}$$

$$0 \to \mathrm{Im}\, \sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_R^n(M, \omega) \to 0, \tag{4.2}$$

where $\sigma_N = \mu\pi$ is the natural epic-monic decomposition of σ_N. Since $\mathrm{Ext}_{S^{\mathrm{op}}}^i(N, \omega) = 0$ for any $1 \le i \le n-2$ and $\mathrm{grade}_\omega\, \mathrm{Ext}_R^{n-1}(M, \omega) \ge n-2$, from the exact sequence (4.1), we have

$$\mathrm{Ext}_{S^{\mathrm{op}}}^i(\mathrm{Im}\, \sigma_N, \omega) = 0$$

for any $1 \le i \le n-2$. Moreover, since $\mathrm{grade}_\omega\, \mathrm{Ext}_R^n(M, \omega) \ge n-1$, from the exact sequence (4.2), we get $\mathrm{Ext}_{S^{\mathrm{op}}}^i(N^{\omega\omega}, \omega) = 0$ for any $1 \le i \le n-2$, which yields

$$\mathrm{Ext}_{S^{\mathrm{op}}}^i([\Omega_{\mathscr{C}}^n(M)]^\omega, \omega) = 0$$

for any $1 \le i \le n-2$. So $\Omega_{\mathscr{C}}^n(M)$ is n-ω-torsionfree by Lemma 2.2.

(1) $\Rightarrow$ (2) By the induction hypothesis, $\mathrm{grade}_\omega\, \mathrm{Ext}_R^{i+1}(M, \omega) \ge i$ for any $1 \le i \le n-2$. So it remains to show that

$$\mathrm{grade}_\omega\, \mathrm{Ext}_R^n(M, \omega) \ge n-1.$$

From the proof of (2) $\Rightarrow$ (1), we have the following facts:

(i) There exists exact sequences (4.1) and (4.2).

(ii) $\Omega_{\mathscr{C}}^n(M) \cong N^\omega$.

(iii) $\mathrm{Ext}_{S^{\mathrm{op}}}^i(N, \omega) = 0$ for any $1 \le i \le n-2$.

(iv) $\mathrm{Ext}_{S^{\mathrm{op}}}^i(\mathrm{Im}\, \sigma_N, \omega) = 0$ for any $1 \le i \le n-2$.

Since $\Omega_{\mathscr{C}}^n(M)$ is n-ω-torsionfree (by (1)) and $\Omega_{\mathscr{C}}^n(M) \cong N^\omega$, N^ω is ω-reflexive and

$$\mathrm{Ext}_{S^{\mathrm{op}}}^i(N^{\omega\omega}, \omega) \cong \mathrm{Ext}_{S^{\mathrm{op}}}^i([\Omega_{\mathscr{C}}^n(M)]^\omega, \omega) = 0$$

for any $1 \le i \le n-2$ by Lemma 2.2. Since $\mathrm{Ext}_{S^{\mathrm{op}}}^i(\mathrm{Im}\, \sigma_N, \omega) = 0$ for any $1 \le i \le n-2$ and we have the exact sequence

$$0 \to \mathrm{Im}\, \sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_R^n(M, \omega) \to 0,$$

$$\mathrm{Ext}_{S^{\mathrm{op}}}^i(\mathrm{Ext}_R^n(M, \omega), \omega) = 0$$

for any $2 \le i \le n-2$. On the other hand, N^ω is ω-reflexive, so $\pi^\omega \mu^\omega = \sigma_N^\omega$ is an isomorphism by [8, Proposition 20.14], and it follows that π^ω and μ^ω are isomorphisms. Moreover, we have a long exact sequence:

$$0 \to [\mathrm{Ext}_R^n(M, \omega)]^\omega \to N^{\omega\omega\omega} \xrightarrow{\mu^\omega} (\mathrm{Im}\, \sigma_N)^\omega \to \mathrm{Ext}_{S^{\mathrm{op}}}^1(\mathrm{Ext}_R^n(M, \omega), \omega) \to \mathrm{Ext}_{S^{\mathrm{op}}}^1(N^{\omega\omega}, \omega) = 0.$$

So

$$[\mathrm{Ext}_R^n(M,\omega)]^\omega \cong \mathrm{Ker}\,\mu^\omega = 0$$

and

$$\mathrm{Ext}^1_{S^{\mathrm{op}}}(\mathrm{Ext}_R^n(M,\omega),\omega) \cong \mathrm{Coker}\,\mu^\omega = 0.$$

Hence we get $\mathrm{grade}_\omega\,\mathrm{Ext}_R^n(M,\omega) \geq n-1$. □

As an immediate consequence of Theorem 4.1, we get the following

Corollary 4.2 *Let* $\mathscr{C}$ *be a full subcategory of* ${}_R^{\perp_n}\omega \cap \mathcal{T}_\omega^n(\mathrm{mod}\,R)$ *containing* $\mathrm{add}_R\,R$. *Then following statements are equivalent for a module* $M \in \mathrm{mod}\,R$.

(1) $\Omega^i_{\mathscr{C}}(M)$ *is* i-ω-*torsionfree for any* $1 \leq i \leq n$.

(2) $\mathrm{grade}_\omega\,\mathrm{Ext}_R^{i+1}(M,\omega) \geq i$ *for any* $1 \leq i \leq n-1$.

In the following result, the equivalence between (1) and (4) is a non-commutative version of [3, Proposition 4.2].

Corollary 4.3 *The following statements are equivalent for a module* $M \in \mathrm{mod}\,R$.

(1) $\Omega^i_R(M)$ *is* i-ω-*torsionfree for any* $1 \leq i \leq n$.

(2) $\Omega^i_{\mathcal{G}_\omega}(M)$ *is* i-ω-*torsionfree for any* $1 \leq i \leq n$.

(3) $\Omega^i_{\overline{\mathrm{add}\,_R\omega}}(M)$ *is* i-ω-*torsionfree for any* $1 \leq i \leq n$.

(4) $\mathrm{grade}_\omega\mathrm{Ext}_R^{i+1}(M,\omega) \geq i$ *for any* $1 \leq i \leq n-1$.

Proof By Corollary 3.5, we have (1) ⇔ (2) ⇔ (3). Putting $\mathscr{C} = \mathrm{add}\,_R R$ in Theorem 4.1, we get (1) ⇔ (4). □

Putting $\mathscr{C} = \mathrm{add}_R\,\omega$ in Theorem 4.1, we get the following

Corollary 4.4 *The following statements are equivalent for a module* $M \in \mathrm{gen}^n(\mathrm{add}\,_R\omega)$.

(1) $\Omega^i_\omega(M)$ *is* i-ω-*torsionfree for any* $1 \leq i \leq n$.

(2) $\mathrm{grade}_\omega\,\mathrm{Ext}_R^{i+1}(M,\omega) \geq i$ *for any* $1 \leq i \leq n-1$.

The following is a special case of Corollary 4.4, which generalizes Theorem 1.4 due to Auslander and Bridger.

Corollary 4.5 *The following statements are equivalent for a module* $M \in \mathrm{mod}\,R$.

(1) $\Omega^i_R(M)$ *is* i-*torsionfree for any* $1 \leq i \leq n-1$.

(2) $\mathrm{grade}_R\,\mathrm{Ext}_R^{i+1}(M,R) \geq i$ *for any* $1 \leq i \leq n-1$.

References

[1] Auslander, M., Bridger, M.: Stable Module Theory. Memoirs Amer. Math. Soc., **94**, Amer. Math. Soc., Providence, RI, 1969

[2] Auslander, M., Buchweitz, R. O.: The homological theory of Cohen–Macaulay approximations. *Soc. Math. France*, **38**, 5–37 (1989)

[3] Takahashi, R.: A new approximation theory which unifies spherical and Cohen–Macaulay approximations. *J. Pure Appl. Algebra*, **208**, 617–634 (2007)

[4] Wakamatsu, T.: On modules with trivial self-extensions. *J. Algebra*, **114**, 106–114 (1988)

[5] Wakamatsu, T.: Tilting modules and Auslander's Gorenstein property. *J. Algebra*, **275**, 3–39 (2004)

[6] Huang, Z. Y.: ω-k-torsionfree modules and ω-left approximation dimension. *Sci. China Ser. A*, **44**, 184–192 (2001)

[7] Huang, Z. Y.: Extension closure of relative syzygy modules. *Sci. China Ser. A*, **46**, 611–620 (2003)

[8] Anderson, F. W., Fuller, K. R.: Rings and Categories of Modules, 2nd edition. In: Grad. Texts in Math., Vol. **13**, Springer-Verlag, Berlin, 1992

[9] Auslander, M., Reiten, I.: Applications of contravariantly finite subcategories. *Adv. Math.*, **86**, 111–152 (1991)

[10] Auslander, M., Reiten, I.: Cohen–Macaulay and Gorenstein artin algebras. In: Representation Theory of Finite Groups and Finite Dimensional Algebras (G. O. Michler, C. M. Ringel, eds.), Bielefeld, 1991, Prog. Math. **95**, Birkhäuser, Basel, 1991, 221–245